Random Process

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RANDOM VARIABLES

- A random variable X is in simplest terms a variable which takes on values at random; it may be thought of as a function of the outcomes of some random experiment.
- The manner of specifying the probability with which different values are taken by the random variable is by the **probability distribution function F(x)**, which is defined by

$$F(x) = P(X \leq x)$$

$$f(x) = \frac{dF(x)}{dx}$$

$$F(A) = \int_{-\infty}^{A} f(x)\, dx$$

RANDOM VARIABLES

- From the definition, the interpretation of f (x) as the density of probability of the event that X takes a value in the vicinity of x is clear.

$$f(x) = \lim_{dx \to 0} \frac{F(x + dx) - F(x)}{dx} = \lim_{dx \to 0} \frac{P(x < X \leq x + dx)}{dx}$$

RANDOM VARIABLES

- The simultaneous consideration of more than one random variable is often necessary or useful. In the case of two, the probability of the occurrence of pairs of values in a given range is prescribed by the ***joint probability distribution function.***

$$F_2(x,y) = P(X \leq x \text{ and } Y \leq y)$$

- where X and Y are the two random variables under consideration. The corresponding***joint probability density function*** is

$$f_2(x,y) = \frac{\partial^2 F_2(x,y)}{\partial x \, \partial y}$$

RANDOM VARIABLES

- For the distribution of X,

$$F(x) = F_2(x,\infty)$$

$$f(x) = \int_{-\infty}^{\infty} f_2(x,y)\,dy$$

RANDOM VARIABLES

- If ***X*** and Y are independent, the event ***X*** <=x is independent of the event Y <=*y;* thus the probability for the joint occurrence of these events is the product of the probabilities for the individual events.

$$\begin{aligned} F_2(x,y) &= P(X \le x \text{ and } Y \le y) \\ &= P(X \le x)P(Y \le y) \\ &= F_X(x)F_Y(y) \end{aligned}$$

The joint probability density function is

$$f_2(x,y) = f_X(x) f_Y(y)$$

Expectations

- The ***expectation*** of a random variable is the sum of all values the random variable may take, each weighted by the probability with which the value is taken.
- This is also called the ***mean value*** of X, or the mean of the distribution of X.
- The expectation of X, which we denote by $\overline{X}$ is

$$\overline{X} = \int_{-\infty}^{\infty} x f(x)\, dx$$

variance

- An important statistical parameter descriptive of the distribution of X is its ***mean-squared value.***

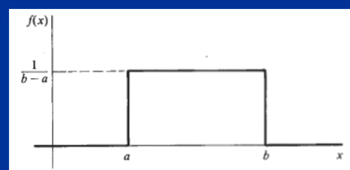
$$\overline{X^2} = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

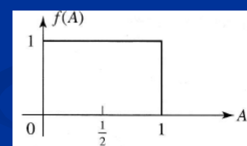
- The ***variance*** of a random variable is the mean-squared deviation of the random variable from its mean; it is denoted by σ^2. The square root of the variance, or σ, is the standard deviation of the random variable.

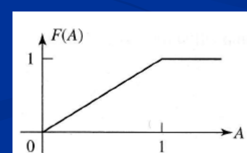
$$\sigma^2 = \int_{-\infty}^{\infty} (x - \overline{X})^2 f(x)\, dx = \overline{X^2} - \overline{X}^2$$

The uniform distributions

- The uniform distribution is characterized by a uniform (constant) probability density over ome finite interval.





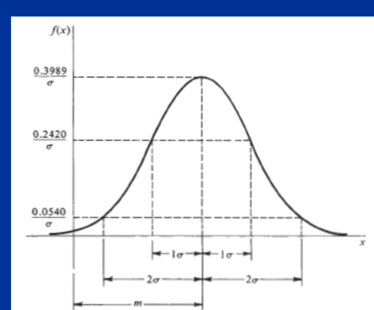


Probability density function f (A) and the probability distribution function F(A) of a uniformly distributed random variable A

Normal Distributions

- The normal probability density function has the analytic form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$



- m = the mean, and σ = the standard deviation.

Random Processes

- Gaussian processes play an important role in communication systems, such as the thermal noise in electronic devices, can be closely modeled by a Gaussian process.
- Definition: A random process X(t) is a Gaussian process if for all n and all (t1, t2, . . . , tn), the random variables $\{X(t_1)\}_{i=1}^{n}$ have a jointly Gaussian density function, which may be expressed as

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}[\det(\boldsymbol{C})]^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{m})^t \boldsymbol{C}^{-1}(\boldsymbol{x}-\boldsymbol{m})\right]$$

- Where the vector x = (x1, x2, ..., xn)^t denotes the n random variables $x_i \equiv X(t_i)$,
- m is the mean value vector, that is, m = E(X), and C is the n x n covariance matrix of the random variables (x1, x2, . . . , xn) with elements

$$c_{ij} = E[(x_i - m_i)(x_j - m_j)]$$

From the above definition it is seen, in particular, that at any time instant /s the random variable X(t0) is Gaussian, and at any two points tl, t2 the random variables (X(t1), X(t2)) are distributed according to a two-dimensional Gaussian random variable.

- Property 1: For Gaussian processes, knowledge of the mean m and covariance C provides a complete statistical description of the process. .
  - Another very important property of a Gaussian process is concerned with its characteristics when passed through a linear time-invariant system.
- Property 2: If the Gaussian process X(t) is passed through a linear time-invariant (LTI) system, the output of the system is also a Gaussian process. The effect of the system on X (t) is simply reflected by a change in the mean value and the covariance of X (t).

Markov Process

- Definition: A Markov process X(t) is a random process whose past has no influence on the future if its present is specified; that is, if $t_n > t_{n-1}$, then

$$P\left[X(t_n) \leq x_n \mid X(t), \quad t \leq t_{n-1}\right] = P\left[X(t_n) \leq x_n \mid X(t_{n-1})\right]$$

- Definition: A Gauss-Markov process X(r) is a Markov process whose probability density function is Gaussian.

- Example, where w_n, is a sequence of zero-mean i.i.d. (white) random variables and p is a parameter that determines the degree of correlation between X_n and X_{n-1};

$$X_n = \rho X_{n-1} + w_n$$

$$E(X_n X_{n-1}) = \rho E(X_{n-1}^2) = \rho \sigma_{n-1}^2$$

- If the sequence w_n is Gaussian, then the resulting process X(t) is a Gauss-Markov process.

Auto-korelasi

- Korelasi x(t) dengan dirinya sendiri disebut auto-korelasi

$$R_x(t) = x(t) \oplus x(t) = \int_{-\infty}^{\infty} x(\tau) x(t+\tau) d\tau$$

- Definition: A random process X (t) is called a white process if it has a flat power spectrum that is, if S_x (f) is a constant for all f.
- if $S_x(f) = C$ for all f ,then

$$\int_{-\infty}^{\infty} S_x(f)\,df = \int_{-\infty}^{\infty} C\,df = \infty$$

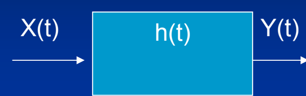
Power Spectrum of Random Processes

- A stationary random process X(t) is characteized in the frequency domain by its power spectrum $S_x(f)$, which is the Fourier transform of the autocorrelation function $R_x(\tau)$ of the random process; that is,

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau}\,dt$$

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f\tau}\,df$$

Linear Filtering



$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t}\, dt$$

$$Y(t) = \int_{-\infty}^{\infty} X(t) h(t-\tau)\, d\tau$$

$$S_y(f) = S_x(f) |H(f)|^2$$